4

A New Equity Condition for Infinite Utility Streams and the Possibility of being Paretian*

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4.1 Introduction

This chapter investigates the properties of a new equity condition for infinite utility streams. The condition, which was introduced in Asheim and Tungodden (2004a), is *Hammond Equity for the Future* (henceforth referred to as **HEF**), and it captures the following ethical intuition: A sacrifice by the present generation leading to a uniform gain for all future generations cannot lead to a less desirable utility stream if the present remains better off than the future even after the sacrifice.

In the terminology of Suzumura and Shinotsuka (2003), this new equity condition is a *consequentialist* condition, in the sense that it expresses preference for a more egalitarian distribution of utilities among generations. In contrast, the 'Weak Anonymity' condition, which often has been invoked to ensure equal treatment of generations (by requiring that any finite permutation of utilities should not change the social evaluation of the stream), is a purely *procedural* equity condition. As we discuss in Asheim and Tungodden (2004a), however, **HEF** is a very weak consequentialist condition. Under certain consistency requirements on the social preferences, it is not only weaker than the ordinary 'Hammond Equity' condition, but it is also implied by other consequentialist equity conditions like the Pigou–Dalton principle of transfers and the Lorenz Domination principle.

From Koopmans (1960), Diamond (1965), and later contributions (e.g., Svensson, 1980; Shinotsuka, 1998; Basu and Mitra, 2003; Fleurbaey

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and Michel, 2003; Sakai, 2003; Xu, 2005) we know that it is problematic in the context of infinite utility streams to combine procedural *equity* conditions with conditions ensuring the *efficiency* of a socially preferred utility stream. In particular, Diamond (1965) states the result that the 'Weak Anonymity' condition cannot be combined with the 'Strong Pareto' condition when the social preferences are complete, transitive and continuous in the sup norm topology (a result that he attributes to M.E. Yaari). This impossibility result has subsequently been strengthened in several ways. The inconsistency remains even if 'Strong Pareto' is replaced by 'Weak Pareto' (Fleurbaey and Michel, 2003) or 'Sensitivity To the Present' (Sakai, 2003). Moreover, Diamond's (1965) proof does not use the full force of the assumption that the social preferences are complete, transitive and continuous in the sup norm topology, and Basu and Mitra (2003a) show that the inconsistency remains even if this assumption is replaced by an assumption of numerical representability.

Suzumura and Shinotsuka (2003) and Sakai (2006) show that the same kind of impossibility results can be established when consequentialist equity conditions are combined with 'Strong Pareto'. In particular, Suzumura and Shinotsuka (2003) establish that the Lorenz Domination principle is not compatible with 'Strong Pareto' when social preferences are upper semi-continuous in the sup norm topology.

The investigations by Suzumura and Shinotsuka (2003) and Sakai (2006) encourage us to carry out a similar analysis for our condition **HEF**. Since **HEF** is a weak condition when compared to other consequentialist equity conditions, it is of interest to establish whether it to a greater extent can be combined with Paretian conditions. We show in this chapter that, unfortunately, this is not the case: Condition **HEF** is not compatible with 'Strong Pareto' when social preferences are upper semi-continuous in the sup norm topology. Both our result and the corresponding result by Suzumura and Shinotsuka (2003) do not require any consistency requirements (like completeness and transitivity) on the social preferences. However, if we impose that the social preferences are complete, transitive and continuous in the sup norm topology, and satisfy an 'Independent Future' condition, then **HEF** cannot even be combined with the 'Weak Pareto' condition. These are discouraging results, given the weakness of **HEF** and its possible ethical appeal.

Our chapter is organized as follows. In section 4.2 we present the setting, and state the conditions that we return to in later sections. In section 4.3 we show under what circumstances **HEF** is implied by other consequentialist equity conditions. In section 4.4 we establish a basic impossibility result, on which the findings in the subsequent sections will be based. In section 4.5 we show that **HEF** cannot be combined with 'Strong Pareto' when preferences satisfy a restricted form for upper semi-continuity in the sup norm topology, while in section 4.6 we report on the inconsistency with 'Weak Pareto' and 'Sensitivity To the Present' under additional conditions. Results relating to the Pigou-Dalton and Lorenz Domination principles are reported as corollaries. In section 4.7 we present examples that serve to clarify the role of the various conditions in the impossibility results that arise in the present framework. Finally, in section 4.8 we discuss what these negative results entail for the usefulness of condition **HEF** and other consequentialist equity conditions as ethical guidelines for intergenerational equity.

4.2 Framework and conditions

Let $\Re$ be the set of real numbers and $\aleph$ the set of positive integers. The set of infinite utility streams is $X = Y^{\aleph}$, where $[0,1] \subseteq Y \subseteq \Re$. Denote by ${}_1\mathbf{u} = (u_1, u_2, \ldots, u_t, \ldots)$ an element of X, where u_t is the utility of generation t, and denote by ${}_1\mathbf{u}_T = (u_1, u_2, \ldots, u_T)$ and ${}_{T+1}\mathbf{u} = (u_{T+1}, u_{T+2}, \ldots)$ the T-head and T-tail of the utility stream respectively. Write ${}_{\text{con}}w = (w, w, \ldots)$ for a stream with a constant level of utility equal to $w \in Y$. Throughout this chapter we assume at least ordinally measurable level comparable utilities; i.e., what Blackorby, Donaldson and Weymark (1984) refer to as 'level-plus comparability'.

For all ${}_1\mathbf{u}$, ${}_1\mathbf{v} \in X$, we write ${}_1\mathbf{u} \geq {}_1\mathbf{v}$ if and only if $u_t \geq v_t$ for all $t \in \aleph$; ${}_1\mathbf{u} > {}_1\mathbf{v}$ if and only if ${}_1\mathbf{u} \geq {}_1\mathbf{v}$ and ${}_1\mathbf{u} \neq {}_1\mathbf{v}$; and ${}_1\mathbf{u} >> {}_1\mathbf{v}$ if and only if $u_t > v_t$ for all $t \in \aleph$.

Social preferences are a binary relation R on X, where for any ${}_1\mathbf{u}$, ${}_1\mathbf{v} \in X$, ${}_1\mathbf{u}\ R\ {}_1\mathbf{v}$ entails that ${}_1\mathbf{u}$ is deemed socially at least as good as ${}_1\mathbf{v}$. Denote by I and P the symmetric and asymmetric parts of R; i.e., ${}_1\mathbf{u}\ I\ {}_1\mathbf{v}$ is equivalent to ${}_1\mathbf{u}\ R\ {}_1\mathbf{v}$ and ${}_1\mathbf{v}\ R\ {}_1\mathbf{u}$ and entails that ${}_1\mathbf{u}$ is deemed socially indifferent to ${}_1\mathbf{v}$, while ${}_1\mathbf{u} P {}_1\mathbf{v}$ is equivalent to ${}_1\mathbf{u}\ R\ {}_1\mathbf{v}$ and $\neg {}_1\mathbf{v}\ R\ {}_1\mathbf{u}$ and entails that ${}_1\mathbf{u}$ is deemed socially preferable to ${}_1\mathbf{v}$. We will consider different sets of conditions on R.

We consider two *consistency* conditions.

Condition O (*Order*) For all ${}_1\mathbf{u}$, ${}_1\mathbf{v} \in X$, ${}_1\mathbf{u}\ R\ {}_1\mathbf{v}$ or ${}_1\mathbf{v}\ R\ {}_1\mathbf{u}$. For all ${}_1\mathbf{u}$, ${}_1\mathbf{v}$, ${}_1\mathbf{w} \in X$, ${}_1\mathbf{u}\ R\ {}_1\mathbf{v}$ and ${}_1\mathbf{v}\ R\ {}_1\mathbf{w}$ imply ${}_1\mathbf{u}\ R\ {}_1\mathbf{w}$.

Condition QT (*Quasi-Transitivity*) For all ${}_1\mathbf{u}$, ${}_1\mathbf{v}$, ${}_1\mathbf{w} \in X$, ${}_1\mathbf{u}\ P\ {}_1\mathbf{v}$ and ${}_1\mathbf{v}\ P\ {}_1\mathbf{w}$ imply ${}_1\mathbf{u}\ P\ {}_1\mathbf{w}$.

Condition **O** implies condition **QT**, while the converse does not hold.

We consider four *continuity* conditions (relative to the sup norm topology). For the results of the present chapter, it is sufficient to use the restricted forms, where we only use the sup norm to compare with streams that are eventually constant. Such restricted continuity conditions are less demanding than their non-restricted counterparts.

Condition C (*Continuity*) For all ${}_1\mathbf{u}$, ${}_1\mathbf{v} \in X$, if $\lim_{n\to\infty} \sup_t |u_t^n - u_t| = 0$ with, for all n, $\neg {}_1\mathbf{v}\ P\ {}_1\mathbf{u}^n$ (resp. $\neg {}_1\mathbf{u}^n\ P\ {}_1\mathbf{v}$), then $\neg {}_1\mathbf{v}\ P\ {}_1\mathbf{u}$ (resp. $\neg {}_1\mathbf{u}\ P\ {}_1\mathbf{v}$).

Condition RC (*Restricted Continuity*) For all ${}_1\mathbf{u}$, ${}_1\mathbf{v} \in X$, if there exists $T \geq 1$ such that $u_t = w$ for all $t \geq T$, and $\lim_{n\to\infty} \sup_t |u_t^n - u_t| = 0$ with, for all n, $\neg {}_1\mathbf{v}\ P\ {}_1\mathbf{u}^n$ (resp. $\neg {}_1\mathbf{u}^n P\ {}_1\mathbf{v}$), then $\neg {}_1\mathbf{v}\ P\ {}_1\mathbf{u}$ (resp. $\neg {}_1\mathbf{u}\ P\ {}_1\mathbf{v}$).

Condition USC (*Upper Semi-Continuity*) For all ${}_1\mathbf{u}$, ${}_1\mathbf{v} \in X$, if $\lim_{n\to\infty} \sup_t |u_t^n - u_t| = 0$ with, for all n, $\neg {}_1\mathbf{v}\ P\ {}_1\mathbf{u}^n$, then $\neg {}_1\mathbf{v}\ P\ {}_1\mathbf{u}$.

Condition RUSC (*Restricted Upper Semi-Continuity*) For all ${}_1\mathbf{u}$, ${}_1\mathbf{v} \in X$, if there exists $T \geq 1$ such that $u_t = w$ for all $t \geq T$, and $\lim_{n\to\infty} \sup_t |u_t^n - u_t| = 0$ with, for all n, $\neg {}_1\mathbf{v}\ P\ {}_1\mathbf{u}^n$, then $\neg {}_1\mathbf{v}\ P\ {}_1\mathbf{u}$.

Condition **C** implies conditions **RC** and **USC**, while each of the latter implies condition **RUSC**. The converses do not hold.

We consider eight *efficiency* conditions. The first four are *Paretian* conditions, where condition **WD** has been analyzed by Basu and Mitra (2007b), while condition **RWP** is used by Asheim and Tungodden (2004a) (but referred to there as ‘Sensitivity’).

Condition SP (*Strong Pareto*) For all ${}_1\mathbf{u}$, ${}_1\mathbf{v} \in X$, if ${}_1\mathbf{u} > {}_1\mathbf{v}$, then ${}_1\mathbf{u}\ P\ {}_1\mathbf{v}$.

Condition WD (*Weak Dominance*) For all ${}_1\mathbf{u}$, ${}_1\mathbf{v} \in X$, if there exists $s \geq 1$ such that $u_s > v_s$ and $u_t = v_t$ for $t \neq s$, then ${}_1\mathbf{u}\ P\ {}_1\mathbf{v}$.

Condition WP (*Weak Pareto*) For all ${}_1\mathbf{u}$, ${}_1\mathbf{v} \in X$, if ${}_1\mathbf{u} >> {}_1\mathbf{v}$, then ${}_1\mathbf{u}\ P\ {}_1\mathbf{v}$.

Condition RWP (*Restricted Weak Pareto*) For all ${}_1\mathbf{u}$, ${}_1\mathbf{v} \in X$, if ${}_1\mathbf{u} >> {}_1\mathbf{v}$ and there exists $T \geq 1$ such that $u_t = w$ and $v_t = x$ for all $t \geq T$, then ${}_1\mathbf{u}\ P\ {}_1\mathbf{v}$.

Condition **SP** implies conditions **WD** and **WP**, while the converses do not hold. Moreover, condition **WP** implies condition **RWP**, while the converse does not hold.

The remaining four are *sensitivity* conditions, where condition **STP** has been analyzed by Sakai (2003), while condition **WS** coincides with Koopmans’ (1960) postulate 2.

Condition SS (*Strong Sensitivity*) For all ${}_2\mathbf{w} \in X$, there exist u_1, $v_1 \in Y$ with $u_1 > v_1$ such that $(u_1, {}_2\mathbf{w})\ P(v_1, {}_2\mathbf{w})$.

Condition STP (*Sensitivity To the Present*) For all ${}_1\mathbf{w} \in X$, there exist ${}_1\mathbf{u}$, ${}_1\mathbf{v} \in X$, and $T \geq 1$ such that $({}_1\mathbf{u}_T, {}_{T+1}\mathbf{w})\ P({}_1\mathbf{v}_T, {}_{T+1}\mathbf{w})$.

Condition RS (*Restricted Sensitivity*) There exist u, $v \in Y$ with $u > v$ such that $(u, {}_{\text{con}}v)P(v, {}_{\text{con}}v)$.

Condition WS (*Weak Sensitivity*) There exist u_1, $v_1 \in Y$ and ${}_2\mathbf{w} \in X$ such that $(u_1, {}_2\mathbf{w})\ P(v_1, {}_2\mathbf{w})$.

Condition **WD** implies condition **SS**, which in turn implies conditions **STP** and **RS**, while the converses do not hold. Furthermore, condition **RS** implies condition **WS**, while the converse does not hold.

Finally, we consider four consequentialist *equity* conditions. The two first require only, as we assume throughout this chapter, at least ordinally measurable level comparable utilities. For complete social preferences these conditions coincide with those suggested by Hammond (1976) and Asheim and Tungodden (2004a), respectively.

Condition HE (*Hammond Equity*) For all ${}_1\mathbf{u}$, ${}_1\mathbf{v} \in X$, if ${}_1\mathbf{u}$ and ${}_1\mathbf{v}$ satisfy that there exists a pair r and s such that $u_r > v_r > v_s > u_s$ and $v_t = u_t$ for $t \neq r, s$, then $\neg {}_1\mathbf{u} \; P \; {}_1\mathbf{v}$.

Condition HEF (*Hammond Equity for the Future*) For all ${}_1\mathbf{u}$, ${}_1\mathbf{v} \in X$, if ${}_1\mathbf{u}$ and ${}_1\mathbf{v}$ satisfy that $u_1 > v_1 > \mathrm{x} > w$ and $u_t = w$ and $v_t = \mathrm{x}$ for all $t > 1$, then $\neg {}_1\mathbf{u} \; P \; {}_1\mathbf{v}$.

The two next equity conditions require, in addition, that utilities are at least cardinally measurable and unit comparable. Such consequentialist equity conditions have been used in the context of infinite streams by, e.g., Birchenhall and Grout (1979), Asheim (1991), and Fleurbaey and Michel (2001), as well as Suzumura and Shinotsuka (2003) and Sakai (2006). The former of the two conditions below is in the exact form suggested by Suzumura and Shinotsuka (2003).

Condition WLD (*Weak Lorenz Domination*) For all ${}_1\mathbf{u}$, ${}_1\mathbf{v} \in X$, if ${}_1\mathbf{u}$ and ${}_1\mathbf{v}$ satisfy that there exist $T > 1$ such that ${}_1\mathbf{v}_{\mathrm{T}}$ Lorenz dominates ${}_1\mathbf{u}_{\mathrm{T}}$ and ${}_{T+1}\mathbf{u} = {}_{T+1}\mathbf{v}$, then $\neg {}_1\mathbf{u} \; P \; {}_1\mathbf{v}$.

Condition WPD (*Weak Pigou-Dalton*) For all ${}_1\mathbf{u}$, ${}_1\mathbf{v} \in X$, if ${}_1\mathbf{u}$ and ${}_1\mathbf{v}$ satisfy that there exist a positive number ε and a pair r and s such that $u_r - \varepsilon = v_r \geq v_s = u_s + \varepsilon$ and $v_t = u_t$ for $t \neq r, s$, then $\neg {}_1\mathbf{u} \; P \; {}_1\mathbf{v}$.

Condition **WLD** implies condition **WPD**, while the converse does not hold. The implications between condition **HEF**, on the one hand, and the three other equity conditions, on the other hand, are treated in the next section. All results in this chapter would still hold if we replaced condition **WPD** by a weaker rank-preserving version where the premise requires also that, for $t \neq r, s$, $v_r \geq v_t$ if $u_r \geq u_t$ and $v_t \geq v_s$ if $u_t \geq u_s$ (cf. Fields and Fei, 1978).

We end this section by stating a condition which is implied by Koopmans' (1960) postulates 3b and 4. It means that a decision concerning only generations from the second period on can be made as if the present time (period 1) was actually at period 2; i.e., as if generations $\{1, 2, \ldots\}$ would have taken the place of generations $\{2, 3, \ldots\}$. It is stated by this name, but in a slightly stronger form, by Fleurbaey and Michel (2003).

Condition IF (*Independent Future*) For all ${}_1\mathbf{u}$, ${}_1\mathbf{v} \in X$ with $u_1 = v_1$, ${}_1\mathbf{u} \; R_1 \; \mathbf{v}$ if and only if ${}_2\mathbf{u} \; R_2 \; \mathbf{v}$.

4.3 Hammond *Equity for the Future*

For streams where utility is constant from the second period on, condition **HEF** states the following: If the present is better off than the future and a sacrifice now leads to a uniform gain for all future generations, then such a transfer from the present to the future cannot lead to a stream that is less desirable in social evaluation, as long as the present remains better off than the future.

To appreciate the weakness of condition **HEF**, consider the following result.

Proposition 1 *Let* $Y \supseteq [0, 1]$. *If* ***QT*** *and* ***RWP*** *hold, then each of* ***HE*** *and* ***WLD*** *implies* ***HEF***. *If* ***O*** *and* ***RWP*** *hold, then* ***WPD*** *implies* ***HEF***.

Proof. Assume $u'' > u' > w' > w''$. We must show under the given conditions that each of **HE**, **WLD**, and **WPD** implies $\neg(u'',_{\text{con}} w'')\ P\ (u',\ _{\text{con}}w')$.

Since $u'' > u' > w' > w''$, there exists an integer $T \geq 1$ and utilities $v,\ x \in Y$ satisfying $u'' > u' > v \geq w' > x > w''$ and $u'' - v = T(x - w'')$.

If **HE** holds, then $\neg(u'',\ _{\text{con}}w'')\ P\ (v,\ x,\ _{\text{con}}w'')$, and by **RWP**, $(u',\ _{\text{con}}w')$ $P\ (v,\ x,\ _{\text{con}}w'')$. By **QT**, $\neg(u'',\ _{\text{con}}w'')\ P\ (u',\ _{\text{con}}w')$.

Consider next **WLD** and **WPD**. Let $_1\mathbf{u}^0 = (u'',\ _{\text{con}}w'')$, and define, for $n \in \{1, \ldots, T\}$, $_1\mathbf{u}^n$ inductively as follows:

$$
\begin{aligned}
u_t^n &= u_t^{n-1} - (x - w'') && \text{for } t = 1\\
u_t^n &= x && \text{for } t = 1 + n\\
u_t^n &= u_t^{n-1} && \text{for } t \neq 1, 1 + n.
\end{aligned}
$$

If **WLD** holds, then $\neg{}_1\mathbf{u}^0\ P\ {}_1\mathbf{u}^T$, and by **RWP**, $(u',\ _{\text{con}}w')\ P\ {}_1\mathbf{u}^T$. By **QT**, $\neg(u'',\ _{\text{con}}w'')\ P\ (u',\ _{\text{con}}w')$ since $_1\mathbf{u}^0 = (u'',\ _{\text{con}}w'')$.

If **WPD** holds, then by **O**, for $n \in \{1, \ldots, T\}$, $_1\mathbf{u}^n\ R\ {}_1\mathbf{u}^{n-1}$, and by **RWP**, $(u',\ _{\text{con}}w')\ P\ {}_1\mathbf{u}^T$. By **O**, $(u',\ _{\text{con}}w')\ P\ (u'',\ _{\text{con}}w'')$ since $_1\mathbf{u}^0 = (u'',\ _{\text{con}}w'')$. Hence, $\neg(u'',\ _{\text{con}}w'')\ P\ (u',\ _{\text{con}}w')$. □

Note that condition **HEF** involves a comparison between a sacrifice by a single generation and a uniform gain for each member of an infinite set of generations that are worse off. Hence, contrary to the standard 'Hammond Equity' condition, if utilities are made (at least) cardinally measurable and fully comparable, then the transfer from the better-off present to the worse-off future specified in condition **HEF** increases the sum of utilities obtained by summing the utilities of a sufficiently large number T of generations. This entails that condition **HEF** is implied by both the Pigou–Dalton principle of transfers and the Lorenz Domination principle, independently of what specific cardinal utility scale is imposed (provided that the consistency conditions specified in Proposition 1 are satisfied). Hence, 'Hammond Equity for the Future' can be endorsed both from an egalitarian and utilitarian point of view. In particular, condition **HEF** is much weaker and more compelling than the standard 'Hammond Equity' condition.

4.4 Basic impossibility result

In the present section we establish that **HEF** is in direct conflict with **RS** under **RUSC**. Hence, there are no restricted sensitive and restricted upper semi-continuous social preferences that satisfy our new equity condition. In the subsequent two sections we note how **RS** is implied by various efficiency conditions. Proposition 2 is thereby used to show how **HEF** cannot be combined with efficiency conditions as long as specific forms of continuity are imposed.

Proposition 2 *Let* $Y \supseteq [0, 1]$. *There are no social preferences satisfying **RUSC**, **RS**, and **HEF**.*

Proof. Suppose there exist social preferences R satisfying **RUSC**, **RS**, and **HEF**.

Step 1: By **RS**, there exists $u, v \in Y$ with $u > v$ such that $(u, {}_{\text{con}}v)\ P\ (v, {}_{\text{con}}v)$. Define $a = u - v$. We claim that there is $b \in (0, a)$ such that

$$(u, {}_{\text{con}}v) P (v + b, {}_{\text{con}}v).$$

If not, for every $b \in (0, a)$ we have $\neg(u, {}_{\text{con}}v)\ P\ (v + b, {}_{\text{con}}v)$. By letting $b \to 0$, we have by **RUSC**: $\neg(u, {}_{\text{con}}v)\ P\ (v, {}_{\text{con}}v)$. This contradicts $(u, {}_{\text{con}}v)\ P\ (v, {}_{\text{con}}v)$ and establishes our claim.

Step 2: For every $c \in (0, b)$, noting that $u > v + b > v + c > v$, **HEF** implies that $\neg(u, {}_{\text{con}}v)\ P\ (v + b, {}_{\text{con}}(v + c))$. By letting $c \to 0$ and using **RUSC**, we get

$$\neg(u, {}_{\text{con}}v) P (v + b, {}_{\text{con}}v).$$

This contradicts the claim proved in Step 1, and establishes the proposition. □

Note that no consistency conditions (like completeness and transitivity) on the social preferences are required for this result

The Diamond–Yaari impossibility result (Diamond, 1965) states that conditions **C** and **SP** are inconsistent with ‘Weak Anonymity’ under the additional assumptions of completeness and transitivity. Actually, the proof provided allows **C** to be replaced by lower semi-continuity and **SP** to be replaced by **WD**, and with a different proof than the one given by Diamond (1965) one can even replace lower semi-continuity by **USC**. Compared to this result, we claim that it is equally worrying that the even weaker conditions **RUSC** and **RS** are inconsistent with assigning priority to an infinite number of worst-off generations in comparisons where the assignment of such priority only reduces the utility of the better-off present generation, as expressed by condition **HEF**. In this respect, note that **HEF** neither implies nor is implied by ‘Weak Anonymity’, and thus Proposition 2 is different from impossibility results based on ‘Weak Anonymity’ as a procedural equity condition.

4.5 Strong Pareto

Since **SP** implies **RS**, it is a straightforward implication of Proposition 2 that **HEF** is in direct conflict with **SP** under **RUSC**. Hence, there are no strongly Paretian and restricted upper semi-continuous social preferences that satisfy 'Hammond Equity for the Future'.

Proposition 3 *Let $Y \supseteq [0, 1]$. There are no social preferences satisfying **RUSC**, **SP**, and **HEF**.*

Since **SP** implies **RWP**, we obtain the following corollary by combining Propositions 1 and 3.

Corollary 1 *Let $Y \supseteq [0, 1]$. If **QT** holds, then there are no social preferences satisfying **RUSC**, **SP**, and **HE**; or **RUSC**, **SP**, and **WLD**. If **O** holds, then there are no social preferences satisfying **RUSC**, **SP**, and **WPD**.*

It should be remarked that the results of Corollary 1 are available in other variants; in particular, it follows from Theorem 3 of Suzumura and Shinotsuka (2003) that condition **QT** is not needed for showing that there are no social preferences satisfying **USC**, **SP**, and **WLD**. Moreover, both Suzumura and Shinotsuka (2003, Theorem 1) and Sakai (2006, Theorem 2) show that only condition **QT** is needed for **USC** and **SP** to be incompatible with a strengthened version of **WPD** (namely, for all ${}_1\mathbf{u}$, ${}_1\mathbf{v} \in X$, if ${}_1\mathbf{u}$ and ${}_1\mathbf{v}$ satisfy that there exist a positive number ε and a pair r and s such that $u_r - \varepsilon = v_r \geq v_s = u_s + \varepsilon$ and $v_t = u_t$ for $t \neq r$, s, then ${}_1\mathbf{v}\ P_1\mathbf{u}$).

4.6 Weaker Paretian conditions

We now show that **HEF** is even in conflict with **WP**, provided that the social preferences satisfy conditions **O**, **RC**, and **IF**. Hence, there are no weakly Paretian, complete, transitive and restricted continuous social preferences that satisfy both 'Independent future' and our new equity condition.

Proposition 4 *Let $Y \supseteq [0, 1]$. If **O** and **IF** hold, then there are no social preferences satisfying **RC**, **WP**, and **HEF**.*

Proposition 4 follows by combining Proposition 2 with the following lemma.

Lemma 1 *Let $Y \supseteq [0, 1]$, and assume that the social preferences R satisfy **O**, **RC**, **WP**, and **IF**. Then the social preferences R satisfy **RS**.*

Proof. Assume that the social preferences R satisfy **O**, **RC**, **WP**, and **IF**. Consider the stream ${}_1\mathbf{u} \in X$ defined by, for all $t \geq 1$, $u_t = 1/t$; i.e.

$${}_1\boldsymbol{u} = \left(1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \ldots\right).$$

By **WP**, ${}_{\text{con}}1\ P\ {}_2\mathbf{u}\ P\ {}_{\text{con}}0$. By **O** and **WP**, there exists $w \in [0,\ 1]$ such that $w = \inf\{x|_{\text{con}}x\ R\ {}_2\mathbf{u}\} = \sup\{x|_2\mathbf{u}\ R\ {}_{\text{con}}x\}$. By **O** and **RC**, ${}_{\text{con}}w\ I\ {}_2\mathbf{u}$ and $w \in (0,\ 1)$. By **IF**, $(1,\ {}_{\text{con}}w)\ I\ {}_1\mathbf{u}$. Since, by **WP**, ${}_1\mathbf{u}\ P\ {}_2\mathbf{u}$, we have that $(1,\ {}_{\text{con}}w)\ I\ {}_1\mathbf{u}\ P\ {}_2\mathbf{u}\ I\ (w,\ {}_{\text{con}}w)$. Hence, by **O**, $(1,\ {}_{\text{con}}w)\ P\ (w,\ {}_{\text{con}}w)$, where $1 > w$. This shows that R satisfies **RS**. □

Since **O** implies **QT** and **WP** implies **RWP**, we obtain the following corollary by combining Propositions 1 and 4.

Corollary 2 *Let* $Y \supseteq [0,\ 1]$. *If* ***O*** *and* ***IF*** *hold, then there are no social preferences satisfying* ***RC****,* ***WP****, and* ***HE****; or* ***RC****,* ***WP****, and* ***WLD****; or* ***RC****,* ***WP****, and* ***WPD****.*

Moreover, as the following proposition establishes, **HEF** is also in conflict with **STP** and **RWP**, provided that the social preferences satisfy conditions **O**, **RC**, and **IF**.

Proposition 5 *Let* $Y = [0,\ 1]$. *If* ***O*** *and* ***IF*** *hold, then there are no social preferences satisfying* ***RC****,* ***STP****,* ***RWP****, and* ***HEF****.*

Proposition 5 follows by combining Proposition 2 with Lemma 3 below. The proof of Lemma 3 makes use of the following result.

Lemma 2 *Let* $Y = [0,\ 1]$, *and assume that the social preferences* R *satisfy* ***O****,* ***RUSC****, and* ***RWP****. Then, for all* ${}_1\mathbf{u} \in X$ *and all* $T \geq 1$, $({}_1\mathbf{u}_T,\ {}_{\text{con}}0)\ R\ {}_{\text{con}}0$.

Proof. Assume that the social preferences R satisfy **O**, **RUSC**, and **RWP**. Let ${}_1\mathbf{u} \in X$. For $a \in (0,\ 1)$, define ${}_1\mathbf{u}(a)$ as follows: $u_t(a) = u_t + a(1 - u_t)$ for $t = 1, \ldots, T$, and $u_t(a) = a$ for $t > T$. For each $a \in (0,\ 1)$, ${}_1\mathbf{u}(a) \in X$, with $u_t(a) \geq a > 0$ for $t = 1, \ldots, T$, and $u_t(a) = a > 0$ for $t > T$. By **RWP**, ${}_1\mathbf{u}(a)\ P\ {}_{\text{con}}0$ for each $a \in (0,\ 1)$. Letting $a \to 0$ and using **O** and **RUSC**, we get $({}_1\mathbf{u}_T,\ {}_{\text{con}}0)$ $R_{\text{con}}0$. □

Lemma 3 *Let* $Y = [0,\ 1]$, *and assume that the social preferences* R *satisfy* ***O****,* ***RC****,* ***STP****,* ***RWP****, and* ***IF****. Then the social preferences* R *satisfy* ***RS****.*

Proof. Suppose that the social preferences R satisfy **O**, **RC**, **STP**, **RWP**, and **IF**, but violate **RS**. Since **RS** is violated, we must have

$$\neg(1,\ {}_{\text{con}}0)P(0,\ {}_{\text{con}}0).$$

Step 1: By **O**, we have $(0,\ {}_{\text{con}}0)\ R\ (1,\ {}_{\text{con}}0)$. On the other hand, by Lemma 2, $(1,\ {}_{\text{con}}0)\ R\ (0,\ {}_{\text{con}}0)$, since **O** and **RC** (and thus, **RUSC**) hold. Hence, we must have

$$(1,\ {}_{\text{con}}0)I(0,\ {}_{\text{con}}0).$$

Define

$$
\begin{aligned}
{}_1\mathbf{x}^0 &= (0, 0, 0, 0, \dots) \\
{}_1\mathbf{x}^1 &= (1, 0, 0, 0, \dots) \\
{}_1\mathbf{x}^2 &= (1, 1, 0, 0, \dots) \\
{}_1\mathbf{x}^3 &= (1, 1, 1, 0, \dots)
\end{aligned}
$$

and so forth. We have already established that ${}_1\mathbf{x}^1 \ I \ {}_1\mathbf{x}^0$. Furthermore, by **IF**, for all $n \in \aleph$, ${}_1\mathbf{x}^n \ I \ {}_1\mathbf{x}^{n-1}$ implies ${}_1\mathbf{x}^{n+1} \ I \ {}_1\mathbf{x}^n$. Since **O** holds, it follows by induction that, for all $n \in \aleph$, ${}_1\mathbf{x}^n \ I \ (0, {}_{\text{con}}0)$.

Step 2: Using **STP**, there exist ${}_1\mathbf{u}$, ${}_1\mathbf{v} \in X$, and $T \geq 1$ such that

$$(u_1, \dots, u_T, {}_{\text{con}}0) \, P \, (v_1, \dots, v_T, {}_{\text{con}}0).$$

By **O** and **RC**, there exists $b \in (0, 1)$ such that

$$(bu_1, \dots, bu_T, {}_{\text{con}}0) \, P \, (v_1, \dots, v_T, {}_{\text{con}}0).$$

For $c \in (0, 1)$, define ${}_1\mathbf{w}(c)$ as follows: $w_t(c) = 1$ for $t = 1, \dots, T$, and $w_t(c) = c$ for $t > T$. Then, by **RWP**, we have ${}_1\mathbf{w}(c) \ P \ (bu_1, \dots, bu_T, {}_{\text{con}}0)$ for each $c \in (0, 1)$. Letting $c \to 0$, and using **O** and **RC**, we have

$${}_1\mathbf{x}^T R (bu_1, \dots, bu_T, {}_{\text{con}}0).$$

On the other hand, by Lemma 2,

$$(v_1, \dots, v_T, {}_{\text{con}}0) \, R \, (0, {}_{\text{con}}0),$$

since **O** and **RC** (and thus, **RUSC**) hold. Hence, ${}_1\mathbf{x}^T \ R \ (bu_1, \dots, bu_T, {}_{\text{con}}0) \ P (v_1, \dots, v_T, {}_{\text{con}}0) \ R \ (0, {}_{\text{con}}0)$, and using **O** we get

$${}_1\mathbf{x}^T \, P \, (0, {}_{\text{con}}0).$$

This contradicts the conclusion reached in Step 1, and establishes the proposition. □

Since **O** implies **QT**, we obtain the following corollary by combining Propositions 1 and 5.

Corollary 3 *Let $Y = [0, 1]$. If **O** and **IF** hold, then there are no social preferences satisfying **RC**, **STP**, **RWP**, and **HE**; or **RC**, **STP**, **RWP**, and **WLD**; or **RC**, **STP**, **RWP**, and **WPD**.*

4.7 Examples

We discuss three examples of social preferences, which clarify the role of the various conditions in the impossibility results arising in the framework of this chapter.

The first example provides an instance of social preferences which satisfy conditions **O**, **RC**, **WP** (and thus, **RWP**), **STP**, and **HEF**. This possibility result points to the critical role played by 'Restricted Sensitivity' (**RS**) in the impossibility result stated in Proposition 2, and the role played by 'Independent Future' (**IF**) in the impossibility results stated in Propositions 4 and 5. The example does not satisfy **RS**, and it does not satisfy **IF** (as is to be expected, since **IF**, in conjunction with the other conditions, would imply that **RS** hold, as we have shown in Lemmas 1 and 3 of this chapter).

The second example provides an instance of *representable* social preferences which satisfy both **RS** and **HEF**. This example does not satisfy **RUSC** since, by Proposition 2, **RS** and **HEF** imply that **RUSC** does not hold. The possibility result that Example 2 constitutes indicates that the even the very weak continuity condition **RUSC**, used in the impossibility result of Proposition 2, is a strong restriction.

The third example provides an instance of social preferences which satisfy conditions **O**, **RC**, **RWP**, **HEF**, and **IF**. This example satisfies neither **WP** nor **STP** since, by Propositions 4 and 5, the other conditions imply that **WP** and **STP** do not hold. Hence, this result illustrates the important role played by **WP** in Proposition 4 and **STP** in Proposition 5.

All three examples indicate that the notion of equity captured by **HEF** is a very weak one. It would be difficult to argue that the social preferences presented in these examples are, in any reasonable sense, 'equitable'. Hence, **HEF** is designed to be a *necessary* condition for equity, classifying as 'inequitable' social preferences that do *not* satisfy the condition.

Example 1 Let $Y \supseteq [0, 1]$, and define, for each ${}_1\mathbf{u} \in X$, $W({}_1\mathbf{u}) = u_2$. Now, define R by

$$\text{for all } {}_1\mathbf{u},\ {}_1\mathbf{v} \in X, {}_1\mathbf{u}\ R_1\mathbf{v} \text{ if and only if } W({}_1\mathbf{u}) \geq W({}_1\mathbf{v}).$$

Hence, the social preferences R are represented by the social welfare function W: $X \to Y$. Then the social preferences R satisfy **O**. They also satisfy **RC** and **WP**. To verify **HEF**, let ${}_1\mathbf{u}, {}_1\mathbf{v} \in X$ satisfy that $u_1 > v_1 > x > w$ and $u_t = w$ and $v_t = x$ for all $t > 1$. Then $W({}_1\mathbf{u}) = w$ and $W({}_1\mathbf{v}) = x$. Thus, $W({}_1\mathbf{v}) > W({}_1\mathbf{u})$, and so ${}_1\mathbf{v}\ P\ {}_1\mathbf{u}$. Finally, one can check that **STP** is satisfied as follows. Given any ${}_1\mathbf{w} \in X$, choose ${}_1\mathbf{u} = {}_{\text{con}}1$ and ${}_1\mathbf{v} = {}_{\text{con}}0$, and $T = 2$. Then we have $({}_1\mathbf{u}_T, {}_{T+1}\mathbf{w}) = (1, 1, {}_{T+1}\mathbf{w})$ and $({}_1\mathbf{v}_T, {}_{T+1}\mathbf{w}) = (0, 0, {}_{T+1}\mathbf{w})$. Thus, $W({}_1\mathbf{u}_T, {}_{T+1}\mathbf{w}) = 1$ and $W({}_1\mathbf{v}_T, {}_{T+1}\mathbf{w}) = 0$, so that $({}_1\mathbf{u}_T, {}_{T+1}\mathbf{w})\ P\ ({}_1\mathbf{v}_T, {}_{T+1}\mathbf{w})$. Clearly, R violates **RS** and **IF**.

Example 2 Let $Y = [0, 1]$, and define, for ${}_1\mathbf{u} \in X$, ${}_1\mathbf{u} \neq {}_{\text{con}}0$, $W({}_1\mathbf{u}) = 1$; and define $W({}_{\text{con}}0) = 0$. Now, define R by

$$\text{for all } {}_1\mathbf{u},\ {}_1\mathbf{v} \in X, {}_1\mathbf{u} R {}_1\mathbf{v} \text{ if and only if } W({}_1\mathbf{u}) \geq W({}_1\mathbf{v}).$$

Hence, the social preferences R are represented by the social welfare function $W\colon X \to \{0, 1\}$. Then the social preferences R satisfy **O**. To verify **HEF**, let ${}_1\mathbf{u}, {}_1\mathbf{v} \in X$ satisfy that $u_1 > v_1 > x > w$ and $u_t = w$ and $v_t = x$ for all $t > 1$. Then $u_1 \neq 0$ so ${}_1\mathbf{u} \neq {}_{\text{con}}0$ and, consequently, $W({}_1\mathbf{u}) = 1$. Also $v_1 \neq 0$ so ${}_1\mathbf{v} \neq {}_{\text{con}}0$ and consequently, $W({}_1\mathbf{v}) = 1$. Then $W({}_1\mathbf{u}) = W({}_1\mathbf{v})$, and so ${}_1\mathbf{v}\ I\ {}_1\mathbf{u}$. To verify **RS**, choose $u = 1$ and $v = 0$. Then $(u, {}_{\text{con}}v) = (1, {}_{\text{con}}0)$ and $(v, {}_{\text{con}}v) = (0, {}_{\text{con}}0)$. Thus, $W(u, {}_{\text{con}}v) = 1$ and $W(v, {}_{\text{con}}v) = 0$ so that $(u, {}_{\text{con}}v)\ P\ (v, {}_{\text{con}}v)$. Clearly, R violates **RUSC**.

Example 3 Let $Y = [0, 1]$, and define, for each ${}_1\mathbf{u} \in X$,

$$W({}_1\mathbf{u}) = \lambda \limsup_{t\to\infty} u_t + (1-\lambda) \liminf_{t\to\infty} u_t, \text{ where } 0 \leq \lambda \leq 1.$$

Now, define R by

$$\text{for all } {}_1\mathbf{u}, {}_1\mathbf{v} \in X, {}_1\mathbf{u} R {}_1\mathbf{v} \text{ if and only if } W({}_1\mathbf{u}) \geq W({}_1\mathbf{v}).$$

Hence, the social preferences R are represented by the social welfare function $W\colon X \to [0, 1]$. If $0 < \lambda < 1$, then the social preferences R presume that utilities are (at least) cardinally measurable and fully comparable. The social preferences R satisfy **O**. They also satisfy **RC** and **RWP**. To verify **HEF**, let ${}_1\mathbf{u}, {}_1\mathbf{v} \in X$ satisfy that $u_1 > v_1 > x > w$ and $u_t = w$ and $v_t = x$ for all $t > 1$. Then $W({}_1\mathbf{u}) = w$ and $W({}_1\mathbf{v}) = x$. Thus, $W({}_1\mathbf{v}) > W({}_1\mathbf{u})$, and so ${}_1\mathbf{v}\ P\ {}_1\mathbf{u}$. To verify **IF**, note that, for all ${}_1\mathbf{u} \in X$, $W({}_1\mathbf{u}) = W({}_2\mathbf{u})$. Hence, ${}_1\mathbf{u}\ R\ {}_1\mathbf{v}$ if and only if ${}_2\mathbf{u}\ R\ {}_2\mathbf{v}$ even if $u_1 = v_1$ does not hold. To see that R violates **WP**, note that ${}_1\mathbf{u}\ I\ {}_{\text{con}}0$ if ${}_1\mathbf{u} \in X$ is defined by, for all $t \geq 1$, $u_t = 1/t$. Clearly, R violates **STP**.

4.8 Concluding remarks

Condition **HEF** assigns priority to an infinite number of worse-off generations in comparisons where the assignment of such priority only reduces the utility of the better-off present generation. We consider this to be a compelling consequentialist equity condition. In particular, as discussed in section 4.3, the condition can be endorsed from both an egalitarian and a utilitarian point of view. It is therefore discouraging that condition **HEF** to such a large extent limits the possibility of being Paretian (cf. Propositions 2, 3, 4, and 5). In principle, there are two ways out of the ethical dilemma that these results pose.

One possibility is to drop continuity. In line with earlier literature, the analysis indicates that continuity conditions are not innocent technical assumptions; rather, such conditions have significant normative implications in the social evaluation of infinite utility streams (e.g., in the words of

Svensson, 1980, p. 1254, 'the continuity requirement is a value judgment'). By employing social preferences over infinite utility streams defined by Basu and Mitra (2007a), Asheim and Tungodden (2004b), and Bossert, Sprumont and Suzumura (2006) (and, if necessary, invoking Szpilrajn's (1930) Lemma to complete the preferences), we can establish the existence of two kinds of social preferences that satisfy **O**, **SP**, **HEF**, and **IF**: One is classical utilitarian, the other is egalitarian and based on leximin. Such preferences are appealing, since they satisfy 'Weak Anonymity' as well as the four consequentialist equity conditions listed in section 4.2. On the other hand, they are all insensitive toward the information provided by either interpersonal level comparability or interpersonal unit comparability. Classical utilitarianism makes no use of interpersonal level comparability (even if utilities are level comparable), while leximin makes no use of interpersonal unit comparability (even if utilities are unit comparable).

Another possibility is to weaken the Paretian requirement to condition **RWP**. Then, as reported in Example 3, there are social preferences satisfying **O**, **RC**, **HEF**, and **IF**. However, the social preferences presented in Example 3 are unappealing, since they entail invariance for the utility during any finite part of the stream. In particular, such social preferences do not satisfy Chichilnisky's (1996) 'No Dictatorship of the Future' condition. However, there are more attractive alternatives. It can be shown that conditions **O**, **RC**, **RWP**, **HEF**, and **IF** imply insensitivity for the interests of the present *only* when the present utility exceeds the stationary equivalent of the utility stream. The conditions do not preclude a trade-off between the interests of the present and future otherwise. Therefore, there exist social preferences satisfying conditions **O**, **RC**, **RWP**, **HEF**, and **IF** that are consistent with both of Chichilnisky's (1996) no-dictatorship conditions ('No Dictatorship of the Present' and 'No Dictatorship of the Future'), and make use of both interpersonal level comparability and interpersonal unit comparability of (at least) cardinally measurable fully comparable utilities. These possibilities are discussed in greater detail in Asheim and Tungodden (2006).

Thus, it is our view that the impossibility results reported in the present chapter should not be used to rule out 'Hammond Equity for the Future' and other consequentialist equity conditions as ethical guidelines for intergenerational equity. They do, however, show that consequentialist equity conditions seriously restrict the set of possible intergenerational social preferences.

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